

Homework 4

Math 121A, Fall 2025.

Due: Wednesday, Sep 24, 11:59 PM

Unless otherwise stated, all solutions require clear, well-written justification. You must explicitly credit any collaborators for each problem.

Problem 1. (*The many wonderful properties of orthogonal matrices*)

The dot product is a special case of an inner product on $\mathbb{R}^n$. Recall that if $x = (x_1 \ \dots \ x_n)^T \in \mathbb{R}^n$ and $y = (y_1 \ \dots \ y_n)^T \in \mathbb{R}^n$ are vectors, then their dot product is

$$\langle x, y \rangle := x \cdot y = \sum_{i=1}^n x_i y_i.$$

We may use this inner product to define the norm

$$\|x\| := \sqrt{\langle x, x \rangle}.$$

We call an $n \times n$ matrix O *orthogonal* if

$$O^T O = \mathbb{1},$$

the $n \times n$ identity matrix.¹

- (a) Show that if O is orthogonal, then so is O^T .
- (b) Show that an orthogonal matrix is invertible (and so defines a change-of-basis matrix).
- (c) We may write the dot product as a matrix multiplication:

$$\langle x, y \rangle = x^T y = (x_1 \ x_2 \ \dots \ x_n) \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}.$$

Show that for an orthogonal matrix O , we have

$$\langle Ox, y \rangle = \langle x, O^T y \rangle.$$

- (d) Show, using properties of the transpose, that $\|Ox\| = \|O^T x\| = \|x\|$ for all $x \in \mathbb{R}^n$. (*Hint: observe that $\|Ox\|^2 = \langle Ox, Ox \rangle$.*)²

¹Unitary matrices are defined by $U^\dagger U = \mathbb{1}$, where $\dagger$ denotes the adjoint with respect to an inner product $\langle x, y \rangle = \bar{x}^T y$ for complex vectors $x, y \in \mathbb{C}^n$. Almost all of these statements have an analogue for unitary matrices.

²Math culture note: we call O an *isometry* with respect to the inner product $\langle x, y \rangle = x \cdot y$, which means O preserves angles and distances.

- (e) Show O is orthogonal if and only if the columns of O , call them

$$O = \begin{pmatrix} | & | & \dots & | \\ v_1 & v_2 & \dots & v_n \\ | & | & \dots & | \end{pmatrix},$$

are orthogonal, meaning

$$\langle v_i, v_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

(Hint: the matrix equation $O^T O = \mathbf{1}$ is really n^2 equations in disguise: what are the (i, j) elements of the identity matrix?). As a consequence, orthogonal matrices are change-of-basis matrices which map orthonormal bases to orthonormal bases.

- (f) Show that $\det(O) \in \{1, -1\}$ for any orthogonal matrix.
 (g) Show that the rotation matrix from last homework,

$$R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad \theta \in [0, 2\pi),$$

is an orthogonal matrix for all θ , and compute its determinant.

- (h) Show that the reflection $R : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, which sends $\hat{k} \mapsto -\hat{k}$ and does nothing to the other basis vectors, is an orthogonal matrix and compute its determinant.

Problem 2. (*Google's most famous algorithm*) Consider the following matrix, which may be interpreted as the probability of a “web surfer” hopping between sites 1 and 2:

$$M = \begin{pmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \end{pmatrix}$$

- (a) What does the spectral theorem tell us about this matrix?
 (b) Diagonalize this matrix, i.e. find an invertible matrix S and a diagonal matrix D such that $M = SDS^{-1}$.
 (c) Compute M^2 using this diagonalization (it is wise to first write $(SDS^{-1})^2$). What do you notice?
 (d) Compute $\lim_{n \rightarrow \infty} M^n$.

Problem 3. (*Our first quantum state, revisited*) Let's continue studying the single-qubit quantum state from last week:

$$\rho = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

- (a) Recall that an orthogonal projection is a linear map such that $\rho = \rho^2 = \rho^\dagger$. Show that ρ is an orthogonal projection.
 (b) By the spectral theorem, ρ has an orthonormal basis of eigenvectors since $\rho = \rho^\dagger$. Diagonalize ρ , i.e. find an orthonormal basis of eigenvectors $\mathcal{B} = \{v_1, v_2\}$ of ρ and find the matrix $[\rho]_{\mathcal{B}}$ in this basis. Compare to the problem from last week.
 (c) Using the diagonal matrix $[\rho]_{\mathcal{B}}$, double-check the claim from part (a) that ρ is an orthogonal projection. Is this calculation more straightforward? Why?

Problem 4. We say that a matrix A is *similar* to B if there is an invertible matrix C such that $A = CBC^{-1}$.

- (a) Show that if A is similar to B , then B is similar to A .
- (b) Show that similar matrices have the same eigenvalues.
- (c) What is the relationship between the eigenvectors of A and those of B ?

Problem 5. (*Function spaces, revisited*) Consider again the vector space of polynomials $p : [0, 1] \rightarrow \mathbb{R}$ given by

$$V = \text{span}\{1, x, x^2\}.$$

We may now equip the (real) vector space of continuous functions $C([0, 1]) = \{v : [0, 1] \rightarrow \mathbb{R} \mid v \text{ is continuous}\}$ with an inner product:

$$\langle v, u \rangle = \int_0^1 v(x)u(x) dx.$$

- (a) Show that this indeed defines an inner product by checking that for any continuous real functions $u, v, w \in C([0, 1])$ and scalars $a, b \in \mathbb{R}$,
 - i. Positive definite: $\langle v, v \rangle \geq 0$, and $\langle v, v \rangle = 0 \iff v = 0$.
 - ii. Linear in the second component: $\langle v, au + bw \rangle = a \langle v, u \rangle + b \langle v, w \rangle$.
 - iii. Symmetric:³ $\langle v, u \rangle = \langle u, v \rangle$.

Note that properties i) and ii) guarantee the inner product is linear in both components.⁴

- (b) Is this basis an orthonormal basis of V ? If not, perform the Gram-Schmidt algorithm to obtain one. (You are welcome to use a computer algebra system, like wolframalpha.com, to compute these integrals.)
- (c) Compute the projection of the (continuous) function $\cos(2\pi x)$ onto V .

Problem 6. (*Spectra and matrix properties*)

The eigenvalues (or “spectrum”) of a matrix contain useful structural information about the linear map it represents.

- (a) Show that a square matrix is not invertible if and only if it has 0 as an eigenvalue.
- (b) Take an $n \times n$ symmetric matrix $M = M^T$. We call M a *contraction* if for every vector $v \in \mathbb{R}^n$, the map M shrinks the size of the vector, i.e.

$$\|Mv\| < \|v\|.$$

Show M is a contraction if and only if the eigenvalues λ of M all satisfy $|\lambda| < 1$. [Hint: use the spectral theorem and the Pythagorean theorem, which states that if \$v_1, v_2, \dots, v_n\$ are orthogonal vectors and \$w_1, w_2, \dots, w_n \in \mathbb{R}\$ then](#)

$$\|w_1v_1 + w_2v_2 + \dots + w_nv_n\|^2 = |w_1|^2 + |w_2|^2 + \dots + |w_n|^2.$$

- (c) Suppose M has an eigenvector v of eigenvalue 1. Interpret in plain English what M does to v .

³When we consider complex functions $f : \mathbb{C} \rightarrow \mathbb{C}$ later, this will be replaced by conjugate symmetry $\langle f, g \rangle = \overline{\langle g, f \rangle}$.

⁴For complex functions, it is antilinear in the first component: $\langle af, g \rangle = \bar{a} \langle f, g \rangle$.